Optimization problems in power series summation

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In this paper we will study a class of slowly converging series. Let

$$c_0, c_1, \ldots \tag{1}$$

be a sequence of numbers. We seek the sum of the power series

$$s(x) = \sum_{r=0}^{\infty} c_r x^r. \tag{2}$$

As is well-known, for certain values of x, this series may converge, but so slowly, that term by term summation is not a feasible method, even if one has access to a supercomputer. We will present methods of determining upper and lower bounds for the sum of the series when the numerical values of the n first terms are given and in addition the general term satisfies an analytic condition. We will treat the important class of the completeley monotonic series, when the bounds for the sum may be determined by solving semi-infinite programs. We extend this process to complex arguments and termine bounds for the real and imaginary parts of the sum. We recall:

Definition 1 *Let the sequence (1) be such that*

$$(-1)^k \Delta^k c_r \geq 0,\ r = 0, 1, \ldots,\ \ k = 0, 1, \ldots \tag{3}$$

where as usual

$$\Delta^0 c_r = c_r,\ \Delta c_r = c_{r+1} - c_r,\ \Delta^{k+1} c_r = \Delta^k c_{r+1} - \Delta^k c_r,\ k = 1, 2, \ldots$$

Then (1) is said to be a completely monotonic sequence and (2) a completely monotonic series.

Example 1 *The following expressions define completely monotonic sequences*

$$c_r = e^{-0.5r},\ c_r = \frac{1}{r+1},\ c_r = \frac{1}{\ln(10+r)},\ r = 0, 1, \ldots \tag{4}$$